ISRAEL JOURNAL OF MATHEMATICS, Vol. 65, No. 3, 1989

ULTRAFILTERS WITH SMALL GENERATING SETS†

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ABSTRACT

It is consistent, relative to ZFC, that the minimum number of subsets of ω generating a non-principal ultrafilter is strictly smaller than the dominating number. In fact, these two numbers can be any two prescribed regular cardinals.

By an *ultrafilter* we mean a nonprincipal ultrafilter on the set ω of natural numbers. A subfamily $\mathscr{G}$ of an ultrafilter $\mathscr{U}$ generates $\mathscr{U}$ if every set $A \in \mathscr{U}$ has a subset $B \in \mathscr{G}$. We shall be interested in the cardinal number

$$u = \min\{ |\mathscr{G}| : \mathscr{G} \text{ generates an ultrafilter}\}$$

and, in particular, in constructing models where u is small. It is clear that $\aleph_1 \leqq u \leqq 2^{\aleph_0}$, and Kunen [3, Ch. VIII, Ex. A10] showed that it is consistent for u to have the minimum possible value, $\aleph_1$, even when $2^{\aleph_0} > \aleph_1$.

To state the more precise results which will concern us here, we introduce two other cardinal numbers, the *bounding number* b, and the *dominating number* d. We say that a function $f: \omega \to \omega$ *dominates* another such function g if, for all but finitely many $n \in \omega$, $f(n) \geqq g(n)$. If this inequality holds for all n, rather than just for all but finitely many, then we say that f *totally dominates* g. Then b is defined to be the minimum cardinality of a family $\mathscr{B}$ of functions such that no function dominates all functions in $\mathscr{B}$, and d is defined to be the minimum cardinality of a family $\mathscr{D}$ of functions such that every function from

† Both authors received partial support from NSF grants.

Received February 24, 1986 and in revised form November 9, 1987

ω to ω is dominated by one in $\mathscr{D}$. (We could use total domination instead of domination in the definition of d but not in the definition of b.) It is well known and easy to see that b is a regular uncountable cardinal and that the cofinality of d is at least b. Solomon [5] showed that $b \leqq u$.

In this paper, we prove the consistency of $u < d$, and in fact we show that there is considerable freedom in the choice of u and d. The consistency of $u < d$ is also established by the models in [1], but the proofs there are considerably more complicated, and they do not allow any freedom in u and d — they have $u = \aleph_1$ and $d = \aleph_2$.

The construction presented here was found by the second author in September 1984, before the one presented in [1]. The first author's contribution was to fill in some details and to write the paper.

THEOREM. *Let ν and δ be uncountable regular cardinals in a model of* ZFC + GCH. *Then there is a countable chain condition forcing extension in which $u = \nu$ and $d = \delta$.*

If $\nu \geqq \delta$, then the desired forcing extension is obtained by first adding δ Cohen reals and then ν random reals. The desired properties of this extension follow routinely from the well-known facts that both Cohen and random forcing satisfy the countable chain condition (c.c.c.), that the ground model is a dominating family in any random extension but not in a Cohen extension, and that neither a random real $a \subseteq \omega$ nor $\omega - a$ includes any infinite set from the ground model. Thus, we shall assume from now on that $\nu < \delta$.

The notion of forcing used to prove the theorem consists of the adjunction of δ Cohen reals followed by a ν-stage iteration of a variant of Mathias forcing relative to carefully chosen ultrafilters. We begin by summarizing the needed facts about this forcing, which was also used in the cited passage in [3].

For any ultrafilter $\mathscr{U}$, $Q(\mathscr{U})$ is the notion of forcing consisting of pairs (a, A) where a is a finite subset of ω, $A \in \mathscr{U}$, and all members of a are smaller than all members of A; (a, A) is an extension of (b, B) if $a \supseteq b$, $A \subseteq B$, and $a - b \subseteq B$. A generic subset G of $Q(\mathscr{U})$ is interdefinable with the subset

$$s(G) = \bigcup \{a : (a, A) \in G\}$$

of ω, which we call the *Mathias real* defined by G. (This terminology is somewhat non-standard. Usually, "Mathias forcing" refers either to a forcing notion analogous to $Q(\mathscr{U})$ but with the second components of conditions being arbitrary infinite sets or to $Q(\mathscr{U})$ for selective ultrafilters $\mathscr{U}$, both of which were considered in [4]. It is convenient to extend the terminology as we have done, but the reader should be warned that some properties of the usual Mathias forcing do not carry over to this more general context. In particular, it will be essential for our proof that, for suitable $\mathcal{U}$, our Mathias reals do not dominate all ground model reals.) It is easy to see that any condition (a, A) forces $s(G)$ to include a and be included in $a \cup A$. (The ordering of $Q(\mathcal{U})$ was designed to correspond to this "information about s in (a, A)".) It follows that $s(G)$ is almost included (i.e. included modulo a finite set) in every $A \in \mathcal{U}$.

We shall say that a finite set $a \subseteq \omega$ is *permitted* by a condition (b, B) if $a \supseteq b$ and $a - b \subseteq B$. This is equivalent to saying that a is the first component of a condition extending (b, B), as we can take the second component to be a suitable final segment of B.

We can now describe our forcing construction in somewhat more detail. Starting with the given model V of ZFC + GCH, we first adjoin a δ-sequence $(r_\alpha : \alpha < \delta)$ of mutually Cohen-generic functions $r_\alpha : \omega \to \omega$. Then we adjoin a ν-sequence $(s_\xi : \xi < \nu)$ of reals by means of a ν-step finite-support iteration. Each s_ξ is a Mathias real over the previous model $V[(r_\alpha : \alpha < \delta)][(s_\eta : \eta < \xi)]$, with respect to a certain ultrafilter $\mathcal{U}_\xi$ in this model. $\mathcal{U}_\xi$ will be constructed (carefully) later, but for the time being we impose on it only the following constraint. Suppose that the sequence $(s_\eta : \eta < \xi)$ of Mathias reals adjoined at earlier steps is almost decreasing. Then we require $\mathcal{U}_\xi$ to contain all these s_η. (Clearly such a $\mathcal{U}_\xi$ will exist.) It then follows that s_ξ is almost included in s_η for each $\eta < \xi$, so our supposition remains true at $\xi + 1$. It trivially remains true at limit ordinals, so we find that the whole sequence $(s_\xi ; \xi < \nu)$ is almost decreasing.

Before continuing the proof, we introduce notation for the various intermediate models to which we shall refer. Define, for $\alpha \leqq \delta$ and $\xi \leqq \nu$,

$$V(\alpha, \xi) = V[(r_\beta : \beta < \alpha)][(s_\eta : \eta < \xi)].$$

Thus, $V(0, 0)$ is the ground model, $V(\delta, 0)$ is the initial Cohen extension, and $V(\delta, \xi)$ is the model obtained after ξ stages in the Mathias forcing iteration. (For $\alpha < \delta$, we do not (yet) know that the s_η's can be obtained by iterated Mathias forcing over $V(\alpha, 0)$.)

Because the iteration is done with finite supports and the c.c.c. holds for Cohen and Mathias forcing, it holds for the entire forcing [6]. Since ν is regular, it follows that each subset of ω (or of any ordinal $< \nu$) in the final extension $V(\delta, \nu)$ is already in $V(\delta, \xi)$ for some $\xi < \nu$. If $A \subseteq \omega$ and $A \in V(\delta, \xi)$, then either A or $\omega - A$ is in $\mathcal{U}_\xi$ (as this is an ultrafilter in $V(\delta, \xi)$) and therefore almost includes s_ξ. Thus, the sets s_ξ ($\xi < \nu$), together with their intersections with cofinite subsets of ω, generate an ultrafilter, namely $\bigcup_{\xi<\nu}\mathscr{U}_\xi$, in $V(\delta, \nu)$. Therefore, this model satisfies $u \leqq \nu$.

In fact, it satisfies $u = \nu$. To see this, we consider an arbitrary family $\mathscr{G}$ of fewer than ν infinite subsets of ω in $V(\delta, \nu)$ and show that it fails to generate an ultrafilter. Since $\mathscr{G}$ can be coded by a subset of an ordinal $< \nu$, it lies in $V(\delta, \xi)$ for some $\xi < \nu$. Let

$$X = \{n \in \omega : |s_\xi \cap n| \text{ is even}\}.$$

(We use the points of s_ξ to partition ω into blocks; X is the union of the even-numbered blocks.) We show that neither X nor its complement $\omega - X$ includes any set from $\mathscr{G}$, or indeed any infinite set $Y \in V(\delta, \xi)$. Suppose the contrary. Let $Y \in V(\delta, \xi)$ be an infinite subset of ω, and let (a, A) be a condition in $Q(\mathscr{U}_\xi)$ forcing "$Y \subseteq X$" or forcing "$Y \subseteq \omega - X$" (over $V(\delta, \xi)$). Let m be the smallest member of A, and let y be an element of Y larger than m. Then the two extensions $(a, A - y)$ and $(a, (A - y) \cup \{m\})$ of (a, A), force "$s_\xi \cap y = a$" and "$s_\xi \cap y = a \cup \{m\}$" respectively. So one of them forces "$y \in X$" and the other forces "$y \notin X$". This contradicts the assumption about (a, A), and this contradiction completes the proof that $u = \nu$ in $V(\delta, \nu)$.

Using the regularity of ν and δ, the inequality $\nu < \delta$, and the assumption that GCH holds in the ground model, one finds, by standard arguments for finite-support c.c.c. iterations, that $2^{\aleph_0} = \delta$ in $V(\delta, \nu)$. Thus, trivially, this model satisfies $d \leqq \delta$, and it remains only to prove the converse inequality. We intend to do this by showing that no family of $< \delta$ functions $\omega \to \omega$ can dominate all the Cohen reals r_α. Carrying out this intention will require that we exercise considerable care in the choice of the ultrafilters $\mathscr{U}_\xi$. For example, if we had the misfortune to choose a selective ultrafilter in $V(\delta, 0)$ as $\mathscr{U}_0$, then the function enumerating s_0 would dominate all the reals in $V(\delta, 0)$, in particular all the r_α's. (More generally, Canjar [2] has shown that Mathias forcing with respect to $\mathscr{U}$ introduces a real dominating all ground model reals unless $\mathscr{U}$ is a P-point with no rapid ultrafilter below it in the Rudin–Keisler ordering.) Our objective, therefore, is to choose the $\mathscr{U}_\xi$'s so as to avoid this and similar misfortunes. It is here that our construction deviates from the one in [3], where the ultrafilters could be chosen arbitrarily.

It will be convenient to normalize names (in forcing languages) for reals, i.e., for functions $\omega \to \omega$, as follows. A name f is to consist of a sequence of pairs $((W_n, f_n) : n \in \omega)$ where each W_n is a maximal antichain in the notion of forcing and $f_n : W_n \to \omega$. The intended interpretation is that W_n is a maximal antichain of conditions deciding values for $f(n)$ and, for $p \in W_n$, $f_n(p)$ is the value that p forces $f(n)$ to have. More formally, the denotation, with respect to a generic set G, of such a sequence $((W_n, f_n) : n \in \omega)$ is the function sending each n to $f_n(p_n)$, where p_n is the unique condition in $W_n \cap G$. It is well-known that every name for a real is forced to be equal to one of these normalized names, so we assume henceforth that all names of reals are normalized.

MAIN LEMMA. *Let M be an inner model within a model M' of* ZFC. *Let $\mathcal{U}$ be an ultrafilter in M. Suppose $g : \omega \to \omega$ is in M' and is not dominated by any element of M. Then there exists $\mathcal{U}'$ such that*:

(1) *$\mathcal{U}'$ is an ultrafilter in M', and $\mathcal{U}' \supseteq \mathcal{U}$.*
(2) *Every maximal antichain of $Q(\mathcal{U})$ in M is also a maximal antichain of $Q(\mathcal{U}')$ in M'.*
(3) *If $f \in M$ and f is a $Q(\mathcal{U})$-name for a real, then $\Vdash_{Q(\mathcal{U}')}$ "f does not dominate g".*

Before proving the lemma, we make same clarifying remarks. (1) implies that $Q(\mathcal{U}) \subseteq Q(\mathcal{U}')$. Incompatible conditions in $Q(\mathcal{U})$ are also incompatible in $Q(\mathcal{U}')$; indeed, compatibility of two conditions (a, A) and (b, B) simply means that some finite set c (which can even be taken to be a or b) is permitted by both of them (for then $(c, A \cap B)$ is a common extension), and this description is clearly absolute. Thus, antichains of $Q(\mathcal{U})$ remain antichains of $Q(\mathcal{U}')$, and the import of (2) is that maximality of antichains is preserved.

This preservation implies that, if G' is an M'-generic subset of $Q(\mathcal{U}')$, then $G = G' \cap Q(\mathcal{U})$ is an M-generic subset of $Q(\mathcal{U})$, for it meets every maximal antichain in M of $Q(\mathcal{U})$. The Mathias real for $\mathcal{U}'$ defined by G' and the Mathias real for $\mathcal{U}$ defined by G are the same, because if $(a, A) \in G'$ contributes a to $s(G')$ then $(a, \omega - \min(A)) \in G$ contributes the same a to $s(G)$.

From (1) and (2) it follows that any (normalized) $Q(\mathcal{U})$-name in M for a real is also a $Q(\mathcal{U}')$-name for a real (indeed, for the same real, in the sense that its denotations with respect to the G and G' of the preceding paragraph are the same). So (3) makes sense. (3) is the essential tool for ensuring that the Mathias reals s_ξ that we adjoin do not introduce a small dominating family.

It will be convenient for the proof to observe that (3) is equivalent to the statement (3′) obtained by changing "dominate" to "totally dominate". The reason is that, if f were a counterexample to (3) then we could find a condition $p \in Q(\mathcal{U}')$ and a $k \in \omega$ such that

$$p \Vdash (\forall n \geqq k) f(n) \geqq g(n).$$

Then we could, in M, modify f so that its values at $0, 1, \ldots, k-1$ are forced (by every condition in $Q(\mathcal{U})$) to agree with the corresponding values of g. Because of (2), every condition in $Q(\mathcal{U}')$ forces this modified f to agree with g on arguments $< k$. But then

$$p \Vdash (\forall n) f(n) \geqq g(n)$$

contrary to (3′).

PROOF OF MAIN LEMMA. Consider, for the moment, an arbitrary ultrafilter $\mathcal{U}'$ in M' extending $\mathcal{U}$. We analyze what it means to say that $\mathcal{U}'$ violates (2) or (3′).

A violation of (2) is given by a maximal antichain L of $Q(\mathcal{U})$ in M and a condition $(a, A) \in Q(\mathcal{U}')$ incompatible with all elements of L. This incompatibility means, as remarked above,

No finite set is permitted by both (a, A) and a member of L.

We say that an A with this property is *forbidden* by L and a. (Notice that the definition of "forbidden" doesn't mention $\mathcal{U}'$.)

A violation of (3′) is given by a name $f = ((W_n, f_n) : n \in \omega)$ in M and a condition $(b, B) \in Q(\mathcal{U}')$ such that

$$(b, B) \Vdash (\forall n) f(n) \geqq g(n).$$

In view of the normalization of f, we can express this property as

(b, B) is incompatible with every $p \in W_n$ such that $f_n(p) < g(n)$.

As above, this can be formulated so as not to mention $\mathcal{U}'$. W say that a set B with this property is *forbidden* by f and b.

To prove the lemma, i.e., to avoid all violations of (2) and (3′) (hence also (3)), it therefore suffices to extend $\mathcal{U}$ to an ultrafilter in M' that contains no forbidden sets of either sort. By Zorn's lemma, it suffices to show that no set in $\mathcal{U}$ (in M) is covered by finitely many forbidden sets (in M').

Suppose the contrary. Let $Z \in \mathcal{U}$ be covered by $A_1, \dots, A_k$, $B_1, \dots, B_k$, where each A_i is forbidden by L_i (a maximal antichain of $Q(\mathcal{U})$ in M) and a_i (a finite subset of ω) while each B_i is forbidden by a name $f = ((W_n, f_n) : n \in \omega)$ and b_i (a finite subset of ω). We have assumed for notational simplicity that the numbers of A's and B's are equal (as we could repeat elements in either list) and that the same f is involved in forbidding all k of the B's (as we could replace different f's by their maximum). We can also assume that the A's and B's are pairwise disjoint subsets of Z, as subsets of forbidden sets are forbidden. Let n_0 be the smallest number greater than all elements of the a_i's and b_i's; we can assume $Z \subseteq \omega - n_0$.

CLAIM. *For every $n \in \omega$ there exists $h(n) \in \omega$ such that $h(n) > n$ and, whenever the interval $Z \cap [n, h(n))$ of Z is partitioned into $2k$ pieces, then at least one of the pieces P has both of the following properties.*

(i) *For each $i \leqq k$, there is a (necessarily finite) set $e \subseteq P$ such that $a_i \cup e$ is permitted by L_i.*

(ii) *For each $i \leqq k$ there is a (necessarily finite) set $e \subseteq P$ such that $b_i \cup e$ is permitted by some $p \in W_n$ such that $f_n(p) < h(n)$.*

PROOF OF CLAIM. The claim is clearly absolute; we prove it in M. Suppose n were a counterexample to the claim. Then, by a compactness argument, we obtain a partition of $Z - n$ into $2k$ pieces, none of which has the desired properties, no matter how large we make $h(n)$ at the end of (ii). Being an ultrafilter in M, $\mathcal{U}$ must contain one of the pieces; call that piece P.

Consider any $i \leqq k$. As L_i is a maximal antichain in $Q(\mathcal{U})$, some $p \in L_i$ is compatible with (a_i, P). A common extension has the form $(a_i \cup e, P')$ where e is a finite subset of P and $a_i \cup e$ is permitted by $p \in L_i$. Thus, (i) holds for P.

Consider again any $i \leqq k$. As in the preceding paragraph, the condition $(b_i, P) \in Q(\mathcal{U})$ has an extension $(b_i \cup e, P')$ permitted by some $p \in W_n$. Thus, (ii) holds for P provided $h(n)$ is chosen to be larger than $f_n(p)$.

This contradicts the fact that P does not have the desired properties, so the claim is proved.

Fix an h as in the claim. Since we proved the claim in M, we can take $h \in M$. (But, by absoluteness, the same h works in M'.) For any $n \geqq n_0$, partition $Z \cap [n, h(n))$ into the sets $A_i \cap [n, h(n))$ and $B_i \cap [n, h(n))$. One of the pieces P satisfies (i) and (ii).

Suppose $P = A_i \cap [n, h(n))$. By (i), find $e \subseteq P \subseteq A_i$ such that $a_i \cup e$ is permitted by some $p \in L_i$. As (a_i, A_i) also permits $a_i \cup e$, this contradicts the fact that A_i is forbidden by L_i and a_i. So P cannot be of the form $A_i \cap [n, h(n))$.

Therefore, $P = B_i \cap [n, h(n))$ for some i. By (ii), find $e \subseteq P$ such that $b_i \cup e$ is permitted by some $p \in W_n$ such that $f_n(p) < h(n)$. But f and b_i forbid B_i, so no $p \in W_n$ compatible with (b_i, B_i), in particular not the p in the preceding sentence, can have $f_n(p) < g(n)$. Therefore, $h(n) \geqq g(n)$. Since n can be any natural number $\geqq n_0$, we have shown that g is dominated by the function $h \in M$, contrary to hypothesis. This completes the proof of the main lemma. □

We are now in a position to begin constructing the ultrafilters $\mathcal{U}_\xi$ in the proof of the theorem. Each $\mathscr{U}_\xi$ will be the union of an increasing chain $(\mathscr{U}(\alpha,\xi):\alpha<\delta)$, where $\mathscr{U}(\alpha,\xi)$ is an ultrafilter in the model $V(\alpha,\xi)$.

We begin by constructing $\mathscr{U}_0$: we define ultrafilters $\mathscr{U}(\alpha,0)$ in $V(\alpha,0)$, for $\alpha\leqq\delta$, by induction on α. These ultrafilters are required to satisfy

(1) If $\alpha<\beta$, then $\mathscr{U}(\alpha,0)\subseteq\mathscr{U}(\beta,0)$,

(2) If $\alpha<\beta$, then every maximal antichain of $Q(\mathscr{U}(\alpha,0))$ in $V(\alpha,0)$ is also a maximal antichain of $Q(\mathscr{U}(\beta,0))$ in $V(\beta,0)$.

(3) If $f\in V(\alpha,0)$ is a $Q(\mathscr{U}(\alpha,0))$-name for a real, then $\Vdash_{Q(\mathscr{U}(\alpha+1,0))}$ "f does not dominate r_α".

To begin the induction, let $\mathscr{U}(0,0)$ be any ultrafilter in the ground model $V=V(0,0)$. (Here and in all that follows, arbitrary choices of ultrafilters should be made in a canonical way, using, for example, a fixed well-ordering of (a large enough piece of) the ground model and the well-orderings that it induces in forcing extensions.) At each successor stage $\alpha+1$, apply the main lemma with $M=V(\alpha,0)$, $M'=V(\alpha+1,0)$, $\mathscr{U}=\mathscr{U}(\alpha,0)$ and $g=r_\alpha$. The hypothesis of the lemma is satisfied because r_α, being Cohen-generic over $V(\alpha,0)$, is not dominated by any real in $V(\alpha,0)$. Let $\mathscr{U}(\alpha+1,0)$ be the $\mathscr{U}'$ given by the lemma. At limit stages λ of uncountable cofinality, let $\mathscr{U}(\lambda,0)=\bigcup_{\alpha<\lambda}\mathscr{U}(\alpha,0)$. This is an ultrafilter and satisfies (2) because every real in $V(\lambda,0)$ is already in some earlier $V(\alpha,0)$. It satisfies (1) obviously, and (3) is vacuous at limit stages. At limit stages λ of countable cofinality, we must extend $\bigcup_{\alpha<\lambda}\mathscr{U}(\alpha,0)$ to an ultrafilter $\mathscr{U}(\lambda,0)$ in $V(\lambda,0)$ such that (2) holds when $\beta=\lambda$. ((3) is again vacuous.) As in the proof of the main lemma, we find that an ultrafilter extending $\bigcup_{\alpha<\lambda}\mathscr{U}(\alpha,0)$ violates (2) if and only if it contains a set A that is forbidden by a maximal antichain L (of $Q(\mathscr{U}(\alpha,0))$ in $V(\alpha,0)$ for some $\alpha<\lambda$) and some finite $a\subseteq\omega$. So, by Zorn's lemma, we need only check that no set in any $\mathscr{U}(\alpha,0)$ is covered by finitely many forbidden sets. But if $Z\in\mathscr{U}(\alpha,0)$ were covered by finitely many sets A_i forbidden by maximal antichains L_i of $Q(\mathscr{U}(\alpha_i,0))$ and $a_i\subseteq\omega$, then, letting γ be the largest of α and the α_i's, we would have $Z\in\mathscr{U}(\gamma,0)$ covered by the A_i which are forbidden by the maximal antichains L_i of $Q(\mathscr{U}(\gamma,0))$ and a_i. (We have used induction hypotheses (1) and (2) here.) But we saw in the proof of the main lemma that this is absurd. (It would mean that $\mathscr{U}(\gamma+1,0)$ doesn't exist.)

This completes the inductive definition of $\mathscr{U}(\alpha,0)$ for $\alpha\leqq\delta$. Set $\mathscr{U}_0=\mathscr{U}(\delta,0)$. Let s_0 be a Mathias real for $\mathscr{U}_0$ over $V(\delta,0)$. Using one of the remarks that we made after the statement of the main lemma, we see that, in virtue of (2), s_0 is also a Mathias real for $\mathscr{U}(\alpha,0)$ over $V(\alpha,0)$ for every α. It follows, by (3), that no real in $V(\alpha,1)=V(\alpha,0)[s_0]$ dominates r_α.

This allows us to repeat the construction with 1 in place of 0. Let $\mathcal{U}(0, 1)$ be any ultrafilter in $V(0, 1) = V[s_0]$ containing s_0. Extend it by using the main lemma at successor stages, taking unions at limit stages of uncountable cofinality, and extending unions so as to preserve maximality of antichains at limit stages of countable cofinality. This defines $\mathcal{U}(\alpha, 1)$ for $\alpha \leqq \delta$, with properties analogous to those of $\mathcal{U}(\alpha, 0)$. Let $\mathcal{U}_1 = \mathcal{U}(\delta, 1)$, and let s_1 be a Mathias real for $\mathcal{U}_1$ over $V(\delta, 1)$. Thus, s_1 is almost included in s_0, since $s_0 \in \mathcal{U}(0, 1) \subseteq \mathcal{U}_1$. Just as for s_0, we see that s_1 is also a Mathias real over each $V(\alpha, 1)$, $\alpha \leqq \delta$, with respect to the ultrafilter $\mathcal{U}(\alpha, 1)$. Thus, $V(\alpha, 2)$ is a two step iteration of Mathias forcing over $V(\alpha, 0)$.

We can clearly repeat this process inductively, obtaining for all finite n and all $\alpha \leqq \delta$ ultrafilters $\mathcal{U}(\alpha, n)$ and Mathias reals s_n for $\mathcal{U}_n = \mathcal{U}(\delta, n)$ over $V(\delta, n)$. The properties (1), (2), and (3), with 0 changed to n, hold for each n. Hence, for each $\alpha \leqq \delta$ and each $n < \omega$, $V(\alpha, n)$ is obtained from $V(\alpha, 0)$ by an n-step iteration of Mathias forcing with respect to the ultrafilters $\mathcal{U}(\alpha, i)$, $i < n$.

Although we have described the construction so far as the successive adjunction of the reals s_n to models $V(\delta, n)$ to produce $V(\delta, n + 1)$, it is clear that we could have formulated all this as the definition of a forcing iteration $T(\delta, n)$ in $V(\delta, 0)$. Our description of $\mathcal{U}(\delta, n)$ in $V(\delta, n)$ actually provides canonical $T(\delta, n)$-names for $\mathcal{U}(\delta, n)$ and thus for $Q(\mathcal{U}(\delta, n))$. Using the latter name $\underset{\sim}{Q}(\delta, n)$, we can define $T(\delta, n + 1)$ as $T(\delta, n) * \underset{\sim}{Q}(\delta, n)$.

We could similarly define, for each $\alpha < \delta$, notions of forcing $T(\alpha, n)$ for iteratively adding n Mathias reals to $V(\alpha, 0)$ using the ultrafilters $\mathcal{U}(\alpha, i)$ (or rather, when $i > 0$, canonical names for these ultrafilters). Our earlier observation that the sequence of Mathias reals $\langle s_i : i < n \rangle$ is not only $T(\delta, n)$-generic over $V(\delta, 0)$ but also $T(\alpha, n)$-generic over $V(\alpha, 0)$ for $\alpha < \delta$ amounts (since the generic sets were chosen arbitrarily) to the observation that every dense subset of $T(\alpha, n)$ in $V(\alpha, 0)$ is also predense (i.e., its closure under extensions is dense) in $T(\delta, n)$. This preservation of density can, of course, also be verified by working directly with the notions of forcing rather than their generic subsets; one proceeds by induction on n, using property (2) of the ultrafilters $\mathcal{U}(\alpha, 0)$ and the analogous property of the higher $\mathcal{U}(\alpha, n)$'s.

We let $T(\alpha, \omega)$, for each $\alpha \leqq \delta$, be the direct limit of the $T(\alpha, n)$'s for $n < \omega$; that is, we iterate the forcing with finite support. Our primary interest is, of course, in $T(\delta, \omega)$, but we shall need to know that this forcing is appropriately related to the $T(\alpha, \omega)$'s for $\alpha < \delta$. Specifically, since $T(\alpha, n) \subseteq T(\delta, n)$ for all finite n, we have $T(\alpha, \omega) \subseteq T(\delta, \omega)$, and we want to know that the density preservation property, noticed in the preceding paragraph for each $n < \omega$, continues to hold for ω.

To see this, let D be any dense subset of $T(\alpha, \omega)$ in $V(\alpha, 0)$, and let $r \in T(\delta, \omega)$. By definition of direct limit, we have $r \in T(\delta, n)$ for some $n < \omega$, and we fix such an n. Then, by well-known properties of finite support iterations, we can identify $T(\delta, \omega)$ with $T(\delta, n) * \underset{\sim}{R}$ for a certain name $\underset{\sim}{R}$ of a notion of forcing, and we can similarly identify $T(\alpha, \omega)$ with $T(\alpha, n) * \underset{\sim}{R}'$ for a certain $\underset{\sim}{R}'$. Under these identifications, r corresponds to $\langle r, 1 \rangle$, where 1 denotes the weakest condition in $\underset{\sim}{R}$, and D becomes a set of pairs $\langle p, \underset{\sim}{q} \rangle$, where $p \in T(\alpha, n)$ forces $\underset{\sim}{q} \in \underset{\sim}{R}'$. The set of first components p of these pairs,

$$\bar{D} = \{ p \in T(\alpha, n) \text{: For some } \underset{\sim}{q},\ \langle p, \underset{\sim}{q} \rangle \in D \},$$

is dense in $T(\alpha, n)$. Indeed, for any $p' \in T(\alpha, n)$, the condition $\langle p', 1 \rangle$ in $T(\alpha, \omega)$ has an extension $\langle p, \underset{\sim}{q} \rangle$ in the dense set D, and then p is an extension of p' in $\bar{D}$. By the preservation of density at level n, it follows that $\bar{D}$ is predense in $T(\delta, n)$. In particular, r is compatible with some $p \in \bar{D}$. But then clearly $\langle r, 1 \rangle$ is compatible with some $\langle p, \underset{\sim}{q} \rangle \in D$. This completes the proof that D is predense in $T(\delta, \omega)$.

We have thus shown that the sequence $\langle s_i : i < \omega \rangle$, obtained by a finite-support Mathias iteration over $V(\delta, 0)$, is also obtained by a finite-support Mathias iteration over each $V(\alpha, 0)$.

Having defined the forcing $Q(\mathscr{U}_n)$ and adjoined the corresponding Mathias-generic reals s_n for all $n < \omega$, we have obtained the models $V(\alpha, \omega)$. We wish to continue the construction, defining $\mathscr{U}(\alpha, \omega)$, $\mathscr{U}_\omega$, and s_ω analogously to what we did for finite n. In order to do this, specifically in order to apply the main lemma at successor stages, we need know the following fact.

Lemma. *No real in $V(\alpha, \omega)$ dominates r_α.*

Proof. We pointed out, after introducing s_0, that no real in $V(\alpha, 1)$ dominates r_α; the same argument shows that, for any finite n, no real in $V(\alpha, n)$ dominates r_α. But $V(\alpha, \omega)$ contains reals that are not in $V(\alpha, n)$ for any finite n, and we must prove that these don't dominate r_α either. We work in $V(\delta, \omega)$. It suffices to prove that no real in $V(\alpha, \omega)$ totally dominates r_α, because $V(\alpha, \omega)$ is closed under finite alterations of reals.

For each $\beta \leqq \delta$ and each $\xi \leqq \omega$, the submodel

$$V(\beta, \xi) = V[(r_\gamma : \gamma < \beta)][(s_n : n < \xi)]$$

is obtained from the ground model V by adjoining a generic subset $G(\beta, \xi)$

of a notion of forcing $P(\beta, \xi)$. By our discussion above of the connection between $T(\delta, \omega)$ and $T(\alpha, \omega)$ for $\alpha < \delta$, this forcing $P(\beta, \xi)$ can be described as first adjoining a β-sequence of mutually Cohen-generic reals r_γ and then adjoining, in a finite-support iteration, Mathias reals s_n for certain (names of) ultrafilters $\mathcal{U}(\beta, n)$.

For each $k < \omega$ and each $\beta \leqq \delta$, the forcing $P(\beta, \omega)$ can be viewed as a two-step iteration $P(\beta, k) * \underset{\sim}{R}(\beta, k)$, where the first step adds β Cohen reals and the first k Mathias reals and the second step adds the rest of the first ω Mathias reals. Thus, $V(\beta, \omega)$ is obtained from $V(\beta, k)$ by adjoining a generic subset of $R(\beta, k)$, the denotation of $\underset{\sim}{R}(\beta, k)$ with respect to $G(\beta, k)$. A condition in $R(\beta, k)$ is an ω-sequence in which the jth term is forced, by the preceding terms, to belong to $Q(\mathcal{U}(\beta, k+j))$, i.e. to be a pair (a, A) where a is a finite subset of ω, $A \in \mathcal{U}(\beta, k+j)$, and all elements of a are smaller than all elements of A; in addition, all but finitely many of the terms are forced by the preceding terms to be (ϕ, ω), since the iteration is done with finite supports. It is clear from this description and the fact that $\mathcal{U}(\beta, \xi) \subseteq \mathcal{U}(\gamma, \xi)$ for $\beta \leqq \gamma$ that every condition in $R(\beta, k)$ is also a condition in $R(\gamma, k)$ for every $\gamma \geqq \beta$.

Suppose now that the lemma failed. So there is a real in $V(\alpha, \omega)$ totally dominating r_α. Fix a $P(\alpha, \omega)$-name f, in the ground model, for such a real, and fix a condition $p \in G(\delta, \omega)$ forcing "f totally dominates r_α". Because the Mathias forcing in $G(\delta, \omega)$ is iterated with finite suport, p is in $G(\delta, k)$ for some $k < \omega$. Fix such a k.

Let f' be obtained by partially evaluating the $P(\alpha, \omega)$-name f with respect to $G(\alpha, k)$. That is, f' is the (essentially unique) $R(\alpha, k)$-name in $V(\alpha, k)$ such that its denotation, with respect to any $V(\alpha, k)$-generic $H \subseteq R(\alpha, k)$, is the same as the denotation of f with respect to the V-generic set $G(\alpha, k) * H$ in $P(\alpha, k) * \underset{\sim}{R}(\alpha, k) = P(\alpha, \omega)$.

We assume, as usual, that f' is normalized, $f' = ((W_n, f'_n) : n \in \omega)$. For each $n \in \omega$, let $g(n) = \min\{f'_n(q) : q \in W_n\}$. Notice that $g : \omega \to \omega$ has been defined in $V(\alpha, k)$.

Consider any n, and fix a condition $q \in W_n \subseteq R(\alpha, k)$ such that $f'_n(q) = g(n)$. By our discussion of the notions of forcing $R(\beta, k)$ above, q is also a condition in $R(\delta, k)$. If H is any $V(\delta, k)$-generic subset of $R(\delta, k)$ such that $q \in H$, and if H' is its restriction to $R(\alpha, k)$, then, using subscripts to indicate denotations of names with respect to generic filters, we have

$$\begin{aligned} f_{G(\delta,k)*H}(n) &= f_{G(\alpha,k)*H'}(n) && \text{since } f \text{ is a } P(\alpha, \omega)\text{-name} \\ &= f'_{H'}(n) && \text{by definition of } f' \\ &= g(n) && \text{since } q \in H'. \end{aligned}$$

On the other hand, $G(\delta, k)$ contains p, which forces f to totally dominate r_α, so

$$f_{G(\delta,k)*H}(n) \geqq r_\alpha(n).$$

Therefore, $g(n) \geqq r_\alpha(n)$. Since n was arbitrary, we have shown that r_α is totally dominated by a real g in $V(\alpha, k)$. But, as we pointed out at the beginning of this proof, the construction of the ultrafilters $\mathcal{U}_k$ for $k < \omega$ ensures that no real in any $V(\alpha, k)$ dominates r_α. This contradiction completes the proof of the lemma. □

Given the lemma, we can now define $\mathcal{U}(\alpha, \omega)$ just as we defined $\mathcal{U}(\alpha, k)$ for $k < \omega$. Indeed, the construction that we performed for finite k can be iterated transfinitely as long as we wish. At limit stages, we need a lemma analogous to the one just proved, but the proof carries over without difficulty. (Actually, it is needed only at limit stages of cofinality ω. At limit stages λ of uncountable cofinality, all reals in $V(\alpha, \lambda)$ are in $V(\alpha, \xi)$ with $\xi < \lambda$, so the analogous lemma becomes trivial.) We iterate the construction for ν steps, obtaining a model $V(\delta, \nu)$. The crucial property of the iteration, guaranteed by the lemma at limit stages and by clause (3) in our choice of $\mathcal{U}(\beta, \xi)$ at successor stages, is that for each $\xi \leqq \nu$ and $\alpha < \delta$, no real in $V(\alpha, \xi)$ dominates r_α.

To complete the proof, we need the observation that conditions in $P(\delta, \nu)$ are essentially countable objects; the following lemma is a weak form of this observation, sufficient for our purposes.

Lemma. *Let $\xi \leqq \nu$.*

(a) *Each condition in $P(\delta, \xi)$ is in $P(\alpha, \xi)$ for some $\alpha < \delta$.*

(b) *Each $P(\delta, \xi)$-name for a real is a $P(\alpha, \xi)$-name for some $\alpha < \delta$.*

Proof. Because δ is regular and uncountable and because $P(\delta, \nu)$ satisfies the c.c.c., (a) implies (b). We prove (a) by induction on ξ. For $\xi = 0$, it is immediate from the definition of Cohen forcing. For limit ξ, it is also immediate because, in a finite support iteration, each condition in $P(\delta, \xi)$ is in some earlier $P(\delta, \eta)$. Finally, if $\xi = \eta + 1$ and p is a condition in $P(\delta, \xi)$, then p consists of a $P(\delta, \eta)$-condition q followed by a name for a $Q(\mathcal{U}_\eta)$-condition (a, A). Induction hypothesis (a) implies that $q \in P(\alpha_1, \eta)$, for some $\alpha_1 < \delta$. Induction hypothesis (b) implies that (a, A) is a $P(\alpha_2, \eta)$-name, for some $\alpha_2 < \delta$. Then $p \in P(\alpha, \xi)$ for $\alpha = \max\{\alpha_1, \alpha_2\}$. □

It follows from part (b) of the lemma and the regularity of δ that any family of fewer than δ reals in $V(\delta, \nu)$ is included in $V(\alpha, \nu)$ for some $\alpha < \delta$. By the results immediately preceding the lemma, such a family fails to dominate the corresponding Cohen real r_α. This proves that the dominating number in $V(\delta, \nu)$ is at least δ, so the theorem is proved.

References

1. A. Blass and S. Shelah, *There may be simple $P_{\aleph_1}$ and $P_{\aleph_2}$ points and the Rudin–Keisler ordering may be downward directed*, Ann. Pure Appl. Logic **33** (1987), 213–243.
2. R. M. Canjar, *Mathias forcing which does not add dominating reals*, preprint.
3. K. Kunen, *Set Theory*, North-Holland, Amsterdam, 1980.
4. A. R. D. Mathias, *Happy families*, Ann. Math. Logic **12** (1977), 59–111.
5. R. C. Solomon, *Families of sets and functions*, Czechoslovak Math. J. **27** (1977), 556–559.
6. R. M. Solovay and S. Tennenbaum, *Iterated Cohen extensions and Souslin's problem*, Ann. Math. **94** (1971), 201–245.